

The Finsler Type of Space-time Realization of Deformed Very Special Relativity

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We investigate here all the possible invariant metric functions under the action of various kinds of semi-direct product Poincaré subgroups and their deformed partners. The investigation exhausts the possible theoretical frameworks for the spacetime realization of Cohen-Glashow's very special relativity and the deformation very special relativity approach by Gibbons-Gomis-Pope. Within Finsler-Minkowski type of spacetime, we find that the spacetime emerge a Finsler type of geometry in most cases both for undermed Poincaré subgroup and for deformed one. We give an explanation that the rotation operation should be kept even in a Lorentz violating theory from geometrical view of point. We also find that the admissible geometry for $DTE3b$, $TE(2)$, $ISO(3)$ and $ISO(2,1)$ actually consists of a family in which the metric function vary with a freedom of arbitrary function of the specified combination of variables. The only principle for choosing the correct geometry from the family can only be the dynamical behavior of physics in the spacetime.

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I. INTRODUCTION

The local Lorentz symmetry and CPT invariance is one of the fundamentals of modern physics. The theoretical investigation and experimental examination of Lorentz symmetry have made considerable progress and attracted a lot of attentions since the mid of 1990s [1].

There are many attempts to investigate the possible Lorentz violation from theoretical aspect. Coleman and Glashow consider the case of spacetime translations along with exact rotational symmetry in the rest frame of the cosmic background radiation, but allow small departures from boost invariance in this frame. They developed a perturbative framework to investigate the deviation from Lorentz invariance in which the departure of Lorentz invariance is parametrized in terms of a fixed timelike fourvector or spurion [2]. Colladay and Kostelecky [3] proposed the model incorporating Lorentz and CPT violation extension of the standard model by introducing into the Lagrangian of more general spurion-mediated perturbations, which are sometimes referred to as expectation values of Lorentz tensors following spontaneous Lorentz breaking. It is also argued that large boosts naturally uncover the structure of spacetime at arbitrary small scales and it is unclear how this could be conciliated with the existence of a fundamental scale for the quantum gravitational phenomena, i.e. the Planck scale. The modification of special relativity with an additional fundamental length scale, the Planck scale, is known as doubly special relativity (DSR) [4]. The realization of DSR

can be noncommutative spacetime or the non-linear realization of Poincaré group [5]. The main feature of these realization of DSR involves the deformed dispersion relation which can also leads to Finsler type of spacetime geometry [6].

Because at low energy scales, parity P , charge conjugation C and time reversal T are individually good symmetries of nature while there is evidence of CP violation for higher energies, one may consider the possible failure of Poincaré symmetry at such high energy scales. One theoretical possibility is that the spacetime symmetry of all the observed physical phenomena might be some proper subgroups of the Lorentz group along with the spacetime translations only if these kind of proper subgroups of Poincaré group incorporating with either of the discrete operations P , T CP or CT , can be enlarged to the full Poincaré group. The generic models based on these smaller subgroups are restricted by the principle of Very Special Relativity (VSR), proposed by Cohen and Glashow [7]. Cohen and Glashow argued that the local symmetry of physics might not need to be as large as Lorentz group but its proper subgroup, while the full symmetry restores to Poincaré group when discrete symmetry P , T or CP enters. The Lorentz violation is thus connected with CP violation. Since CP violating effects in nature are small, it is possible that Lorentz-violating effects may be similarly small. They identified these VSR subgroups up to isomorphism as $T(2)$ (2-dimensional translations) with generators $T_1 = K_x + J_y$ and $T_2 = K_y - J_x$, where \mathbf{J} and \mathbf{K} are the generators of rotations and boosts respectively, $E(2)$ (3-parameter Euclidean motion) with generators T_1, T_2 and J_z , $HOM(2)$ (3-parameter orientation preserving transformations) with generators T_1, T_2 and K_z and $SIM(2)$

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(4-parameter similitude group) with generators T_1, T_2, J_z and K_z . The spurion strategy can also be applied to VSP. The invariant tensor for group $E(2)$ can be a 4-vector $n = (1, 0, 0, 1)$ while the symmetry groups $T(2)$ admits many invariant tensors. There is neither invariant tensors for $HOM(2)$ and $SIM(2)$ nor the local Lorentz symmetry breaking perturbative discription for either of these groups.

Concerning how to realize VSR, Sheikh-Jabbar et.al proved that the quantum field theory on the noncommutative Moyal plane with light-like noncommutativity possesses VSR symmetry [8]. For any given QFT on commutative Minkowski space its VSR invariant counterpart is a noncommutative QFT, NCQFT, which is obtained by replacing the usual product of field operators with the nonlocal Moyal $*$ -product. The NCQFT on noncommutative Moyal plane with light-like noncommutativity realization of VSR actually needs to twist deform the VSR subgroup Poincaré group.

Inspiring by the fact that Poincaré group admits the unique deformation into de Sitter group, Gibbons, Gomis and Pope find that the subgroup $ISIM(2)$ considered by Cohen and Glashow admits a 2-parameter family of continuous deformations which may be viewed as a quantum corrections or the quantum gravity effect to the very special relativity, but none of these give rise to noncommutative translations analogous to those of the de Sitter deformation of the Poincaré group: space-time remains flat. Among the 2-parameter family of deformation of $ISIM(2)$, they find that only a 1-parameter $DISIM_b(2)$, the deformation of $SIM(2)$, is physically acceptable [9]. The line element invariant under $DISIM_b(2)$ is Lorentz violating and of Finsler type, $ds^2 = (\eta_{\mu\nu} dx^\mu dx^\nu)^{1-b} (n_\mu dx^\mu)^{2b}$. The $DISIM_b(2)$ invariant action for point particle and the wave equations for spin 0, $\frac{1}{2}$ and 1 are derived in their paper. The equation for spin 0 field is in general a nonlocal equation, since it involves fractional even irrational derivatives.

In our previous paper we follow Gibbons-Gomis-Pope's approach on the deformation of $ISIM(2)$ and investigate the deformation of all such kind of subgroups of Poincaré group which are the semi-product of three generators and four generators Lorentz subgroups with the space-time translation group $T(4)$ (semi-product Poincaré subgroup) and the five dimensional representations, which are inherited from the five dimensional representation of Poincaré group, (the natural representation) of all the semi-product Poincaré subgroup as well as their deformed partners [10]. We find that the deformation of semi-product Poincaré subgroup may have more than one families that are physically acceptable. There may be more than one inequivalent natural representations for one family of deformation of a specific Poincaré subgroup. Usually the deformation of the original Lorentz

subgroup part causes the rotational operation an additional accompanied scale factor which is not reasonable for we believe that the departure from Lorentz symmetry should be from boost rather than rotational operation. Anyhow most deformed boost operations do indeed have an additional accompanied scale factors which will play a key role in the search of group action invariant geometry and construction of field theories in the spactime of the invariant geometry.

In the present paper, we investigate all the possible Finsler geometry realization of spacetime possessing the semi-product Poincaré subgroups and their deformed partner symmetry. To deal with the additional accompanied scale transformation of rotation and boost operation, we find all the independent scale covariant rank 1, 2 and 3 tensors for all cases of the symmetry groups. The existence of invariant metric function automatically excludes the additional accompanied scale transformation of rotation operation which is consistent with our argument that the lorentz invariance should not be broken in rotation but in boost [10]. We find that the admissible invariant metric function contains an arbitrary function of the specified combination of variables freedom, which can not be fixed by the investigation of symmetry. To fix the freedom, it is needed to study the dynamics in the corresponding spacetime. We investigate the dynamics of particles and field theories in our next subsequent pater.

This paper is organized as follows. We first give a brief introduction to Finsler geometry which leads the concept of Finsler-Minkowski spacetime which we concentrate in this paper. Then we give the general methods to construct the invariant metric function under some group action. The most part of the paper is devoted to the seeking of invariant metric functions under the groups we obtained in our previous paper [10]. At last we give a brief discussion of our result and outlook of the investigation on dynamics in our next subsequent paper. We notice that there is also other approach indicating that VSR may be realized by Finsler geometry [11].

II. FINSLER GEOMETRY

Let us start with a brief review of the basic notions relevant for Finsler geometries [12–17].

A. The Metric Structure

In Riemann geometry, the line element is of the form

$$ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \quad (1)$$

where $g_{\mu\nu}$ is the metric tensor of the manifold, which is the function of x^μ . The Finsler geometry is a generaliza-

tion of Riemann geometry with the more general form of line element

$$ds = F(x^\mu, dx^\mu), \quad (2)$$

where F is defined on the tangent bundle of the manifold and a degree 1 homogenous function of dx^μ , which includes the Riemann metric as a special case. More specifically, the Finsler metric norm $F : TM \mapsto \mathbb{R}$ which is a real function $F(x, y)$ of a spacetime point x and of a tangent vector $y \in T_x M$, satisfies

- Regularity: F is smooth on the entire slit tangent bundle $TM \setminus 0$,
- Non-degeneracy: $F(x, y) \neq 0$ if $y \neq 0$,
- Positive homogeneity: $F(x, \lambda y) = |\lambda| F(x, y), \forall \lambda \in \mathbb{R}$.

The so called fundamental tensor can be defined

$$g_{\mu\nu} = \frac{1}{2} \dot{\partial}_\mu \dot{\partial}_\nu F^2, \quad (3)$$

where and hereafter $\dot{\partial}_\mu$ represents $\partial/\partial x^\mu$ and $\dot{\partial}_\mu$ represents $\partial/\partial y^\mu$, which is required to be continuous and non-degenerate. It can be shown that (3) is equivalent to

$$F(x, y) = \sqrt{g_{\mu\nu}(x, y) y^\mu y^\nu}, \quad (4)$$

which shows that $g_{\mu\nu}(x, y)$ is a homogeneous function of degree zero of the vector y . Also, since by definition $g_{\mu\nu}$ is non degenerate, it admits an inverse $g^{\mu\nu}$ such that $g^{\mu\nu}(x, y) g_{\nu\alpha}(x, y) = \delta^\mu_\alpha$, a metric tensor similar to one in Riemann geometry with the difference from Riemann geometry that the metric tensor here does not depend only on coordinates of base manifold but also coordinates of tangent space.

The Finsler metric tensor thus defined must be index symmetric. The derivatives to y^μ are also index symmetric,

$$\dot{\partial}_\alpha g_{\mu\nu} = \dot{\partial}_\mu g_{\nu\alpha} = \dot{\partial}_\nu g_{\alpha\mu}. \quad (5)$$

One can introduce the Christoffel symbols of the first and second kind in terms of Finsler metric tensor. There is a connection between the so called spray induced by F and the Christoffel symbols,

$$G(y)^\mu = \frac{1}{2} \gamma_{\alpha\beta}^\mu y^\alpha y^\beta, \quad (6)$$

where

$$\gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\nu} \left(\dot{\partial}_\alpha g_{\beta\nu} + \dot{\partial}_\beta g_{\alpha\nu} - \dot{\partial}_\nu g_{\alpha\beta} \right) \quad (7)$$

are the formal Christoffel symbols of the second kind.

Apart from the well-known geometric quantities in Riemann geometry, there are many geometric quantities which are non-Riemannian and unique for Finsler geometry, e.g. the Cartan tensor which specifies the departure of the manifold from Riemannian

$$C_{\alpha\beta\sigma} = \frac{1}{2} \dot{\partial}_\sigma g_{\alpha\beta}. \quad (8)$$

It is apparent that the Cartan tensor is full symmetric. The manifold with zero Cartan tensor is Riemannian and viceversa. The length of Cartan co-vector $C_\mu = C_{\mu\alpha\beta} g^{\alpha\beta}$, $C = g^{\mu\nu} C_\mu C_\nu$, can be utilized to describe the departure of a Finsler manifold from Riemannian.

Finsler geometry can be regarded as the geometry of the tangent bundle TM . The local coordinate x^μ of $x \in M$ give rise to the local coordinate of $\{x^\mu, y^\alpha\} \in TM$ through the mechanism

$$y = y^\mu \frac{\partial}{\partial x^\mu} \quad (9)$$

where $y^\mu = y^\mu(x^\alpha)$ are fiberwise global. Under local coordinate transformation $x'^\mu = x'^\mu(x)$, the vector of tangent space transforms like

$$y'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} y^\alpha. \quad (10)$$

So under coordinate transformation

$$\begin{cases} x'^\mu = x'^\mu(x), \\ y'^\mu = y'^\mu(x, y), \end{cases} \quad (11)$$

the coordinate base vectors transform as

$$\begin{cases} \hat{\partial}'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} \hat{\partial}_\alpha + \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} y'^\nu \hat{\partial}_\alpha, \\ \dot{\partial}'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} \dot{\partial}_\alpha. \end{cases} \quad (12)$$

The tangent bundle of the manifold TM has a local coordinate basis that consists of the $\frac{\partial}{\partial x^\mu}$ and the $\frac{\partial}{\partial y^\alpha}$. However, under the transformation on TM induced by a coordinate change on M , the vectors $\frac{\partial}{\partial x^\mu}$ transform in a somewhat complicated manner, while the $\frac{\partial}{\partial y^\alpha}$ do not have this problem. The cotangent bundle of TM has a local coordinate basis $\{dx^\mu, dy^\alpha\}$. Under the induced transformation, the behavior of dx^μ is simple while one of dy^α is not.

The remedy lies in replacing $\frac{\partial}{\partial x^\mu}$ by

$$\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N_\mu^\alpha \frac{\partial}{\partial y^\alpha} \quad (13)$$

and dy^α by

$$\delta y^\alpha = dy^\alpha + N_\mu^\alpha dx^\mu, \quad (14)$$

where

$$N_\mu^v = \frac{1}{2} y^\alpha y^\beta \dot{\partial}_\mu \gamma_{\alpha\beta}^v + \gamma_{\mu\alpha}^v y^\alpha \quad (15)$$

are the Finsler nonlinear connection.

They indeed have simple behavior under transformations induced by coordinate changes on M . Thus the two new natural(local) bases that are dual to each other are

- $\{\frac{\delta}{\delta x^\mu}, F \frac{\partial}{\partial y^\alpha}\}$ for the tangent bundle of $TM \setminus \mathbf{0}$,
- $\{dx^\mu, \delta y^\alpha\}$ for the tangent bundle of $TM \setminus \mathbf{0}$.

Moreover there is a relation between the non-linear connection and the spray

$$N_\nu^\mu = \dot{\partial}_\nu G^\mu. \quad (16)$$

B. The Connection Structure

In Finsler geometry, the covariant derivative need to be carried on two sets of coordinates, so the connection structure is larger than Riemann geometry,

$$\begin{cases} \hat{\nabla}_\mu A^\alpha = \hat{\partial}_\mu A^\alpha + \Gamma_{\beta\mu}^\alpha A^\beta, \\ \hat{\nabla}_\mu A_\alpha = \hat{\partial}_\mu A_\alpha - \Gamma_{\alpha\mu}^\beta A_\beta, \\ \dot{\nabla}_\mu A^\alpha = \dot{\partial}_\mu A^\alpha + \Lambda_{\beta\mu}^\alpha A^\beta, \\ \dot{\nabla}_\mu A_\alpha = \dot{\partial}_\mu A_\alpha - \Lambda_{\alpha\mu}^\beta A_\beta. \end{cases} \quad (17)$$

Similar to Riemann geometry, one can impose the adaptable condition between metric and connection by

$$\hat{\nabla}_\mu g_{\alpha\beta} = \dot{\nabla}_\mu g_{\alpha\beta} = 0. \quad (18)$$

One has

$$\begin{cases} \Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\rho} (\delta_\mu g_{\rho\nu} + \delta_\nu g_{\rho\mu} - \delta_\rho g_{\mu\nu}), \\ \Lambda_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\rho} \dot{\partial}_\rho g_{\mu\nu}, \end{cases} \quad (19)$$

where $\Gamma_{\mu\nu}^\sigma$ is called the Chern connection and the adaptable connection in tangent space is just the Cartan tensor. The triple $(N_\mu^v, \Gamma_{\mu\nu}^\sigma, \Lambda_{\mu\nu}^\sigma)$ is called Cartan connection. The other three popular Finsler connections are

1. Chern-Rund connection $(N_\mu^v, \Gamma_{\alpha\beta}^\mu, 0)$,
2. Berwald connection $(N_\mu^v, G_{\alpha\beta}^\mu, 0)$,
3. Hashiguchi connection $(N_\mu^v, G_{\mu\nu}^\sigma, C_{\mu\nu}^\sigma)$,

where $G_{\mu\nu}^\sigma$ is called Berwald connection defined by $G_{\mu\nu}^\sigma = \dot{\partial}_\nu N_\mu^\sigma = \dot{\partial}_\mu \dot{\partial}_\nu G^\sigma$. Berwald connection and Chern connection are connected each other by a full symmetric Landsberg tensor $L_{\alpha\beta\sigma}$,

$$G_{\mu\nu}^\sigma = \Gamma_{\mu\nu}^\sigma + L_{\mu\nu}^\sigma. \quad (20)$$

In Riemann geometry, torsion is free from the adaption condition of connection while in Finsler geometry, torsion is somewhat inevitable,

$$\begin{cases} [\delta_\mu, \delta_\nu] = R_{\mu\nu}^\alpha \dot{\partial}_\alpha, \\ \left[\begin{matrix} \delta_\mu, \dot{\partial}_\nu \\ \dot{\partial}_\mu, \dot{\partial}_\nu \end{matrix} \right] = G_{\mu\nu}^\alpha \dot{\partial}_\alpha, \\ \left[\begin{matrix} \delta_\mu, \dot{\partial}_\nu \\ \dot{\partial}_\mu, \dot{\partial}_\nu \end{matrix} \right] = 0, \end{cases} \quad (21)$$

where $R_{\mu\nu}^\sigma$ is the Finsler torsion tensor

$$R_{\mu\nu}^\sigma = \delta_\nu N_\mu^\sigma - \delta_\mu N_\nu^\sigma, \quad (22)$$

which does not vanish in Riemann geometry and is just $R_{\mu\nu}^\sigma = y^\alpha R_{\alpha\mu\nu}^\sigma$, where $R_{\alpha\mu\nu}^\sigma$ is the Riemann curvature tensor.

C. Curvature

In Finsler geometry, there are several tensors which can be used to describe the curvature of the manifold.

The Finsler curvature tensor is

$$F_{\sigma\mu\nu}^\rho = \dot{\partial}_\sigma R_{\mu\nu}^\rho, \quad (23)$$

which is just the Riemann curvature tensor in Riemann geometry.

The flag curvature tensor is

$$F_\mu^v = y^\alpha R_{\mu\alpha}^v. \quad (24)$$

The non-Riemannian Berwald curvature tensor is

$$B_{\rho\mu\nu}^\sigma = \dot{\partial}_\rho G_{\mu\nu}^\sigma. \quad (25)$$

The Berwald connection leads to another rank 4 curvature tensor

$$\tilde{R}_{\rho\mu\nu}^\sigma = \delta_\mu G_{\rho\nu}^\sigma - \delta_\nu G_{\rho\mu}^\sigma + G_{\rho\nu}^\gamma G_{\gamma\mu}^\sigma - G_{\rho\mu}^\gamma G_{\gamma\nu}^\sigma. \quad (26)$$

The analogous curvature tensor of Riemann geometry in a Finsler setting for connection $(N_\mu^v, H_{\alpha\beta}^\mu, V_{\alpha\beta}^\mu)$ is

$$\begin{cases} S_{\sigma\alpha\beta}^\mu = \delta_\beta H_{\sigma\alpha}^\mu - \delta_\alpha H_{\sigma\beta}^\mu + H_{\rho\beta}^\mu H_{\sigma\alpha}^\rho - H_{\rho\alpha}^\mu H_{\sigma\beta}^\rho, \\ P_{\sigma\alpha\beta}^\mu = \dot{\partial}_\beta V_{\sigma\alpha}^\mu - \dot{\partial}_\alpha V_{\sigma\beta}^\mu + V_{\rho\beta}^\mu V_{\sigma\alpha}^\rho - V_{\rho\alpha}^\mu V_{\sigma\beta}^\rho, \\ X_{\sigma\alpha\beta}^\mu = \dot{\partial}_\beta H_{\sigma\alpha}^\mu - \delta_\alpha V_{\sigma\beta}^\mu + V_{\rho\beta}^\mu H_{\sigma\alpha}^\rho - H_{\rho\beta}^\mu V_{\sigma\alpha}^\rho, \end{cases} \quad (27)$$

where $S_{\sigma\alpha\beta}^\mu$ is the curvature tensor in the pure horizontal direction, $P_{\sigma\alpha\beta}^\mu$ is the curvature tensor in the pure vertical direction while $X_{\sigma\alpha\beta}^\mu$ is the curvature tensor in the mixed directions.

There is a special class of Finsler manifold which is worthy to pay attention, the Minkowski manifold. In Finsler geometry, Minkowski manifold is a class of flat manifolds

whose Finsler norm does not change with the coordinate on the base manifold and hence a function of the coordinate of the vector space, $F = F(y^\alpha)$. The metric tensor depends only on y^α too. So $\gamma_{\alpha\beta}^\mu = 0$ and the non-linear connection $N_\mu^v = 0$ which lead all the connections of Minkowski manifold to be zero and the zero curvatures. So Finsler-Minkowski manifold is flat. In this paper, we concentrate on the Finsler-Minkowski manifold to seek the invariant metric under semi-direct product Poincaré subgroups and their deformed partners.

III. THE INVARIANT METRIC OF SPACETIME UNDER SEMI-DIRECT PRODUCT POINCARÉ SUBGROUPS AND THEIR DEFORMED PARTNERS

In principle, the spacetime geometry does not have to be Riemann geometry. The reason why it is not some more general type of geometry, e.g. Finsler geometry, but Riemannian in the scheme of general relativity is because of equivalence principle when the local symmetry is Lorentz group. In the scheme of VSR, the local symmetry is not the entire Lorentz group but its proper subgroup. It is not necessary that the general very special relativity has to be Riemannian any more. In fact it is revealed that the general very special relativity is Finslerian under a special deformation $Disim_b(2)$ of $Isim(2)$ symmetry. We have systematically investigate all the possible deformation of Poincaré subalgebra which include all the possible symmetry of very spacial relativity and get their corresponding natural matrix representations already. In this section we will investigate what kind of Finsler geometries that all the possible deformation of Poincaré subalgebra correspond to.

Without losing generality, we assume that the Finlerian metric F^2 consists of M parts factors,

$$F^2 = \prod_{i=1}^M F_i. \quad (28)$$

The F_i has the form

$$F_i = M_i^{E_i}, \quad (29)$$

where E_i is constant and M_i satisfies

$$M_i(y^\mu) = G_{\mu_1\mu_2\ldots\mu_{p_i}} \prod_{j=1}^{p_i} y^{\mu_j}. \quad (30)$$

The $G_{\mu_1\mu_2\ldots\mu_{p_i}}$ is constant tensor. So F_i is a degree $p_i E_i$ homogenous function of tangent space coordinates y_μ . For F^2 is a degree 2 homogenous function of y_μ , we have

$$\sum_{i=1}^M p_i E_i = 2. \quad (31)$$

Suppose T_a is the group element of single parameter Lie group generated by the generator of spacetime symmetry group, we can demand that under the action of T_a , M_i satisfies

$$M_i(T_a(y^\mu)) = A_{ia} M_i(y^\mu). \quad (32)$$

Suppose the action of T_a on the coordinates x^μ of space-time manifold is

$$T_a(x^\mu) = (R_a)^\mu_\alpha x^\alpha + (P_a)^\mu, \quad (33)$$

then the action of T_a on the coordinates y^μ of the tangent space is

$$T_a(y^\mu) = (R_a)^\mu_\alpha y^\alpha. \quad (34)$$

From (30) and (32), we have

$$\prod_{j=1}^{p_i} (R_a)_{\mu_j}^{\alpha_j} G_{\alpha_1\alpha_2\ldots\alpha_{p_i}} = A_{ia} G_{\mu_1\mu_2\ldots\mu_{p_i}}. \quad (35)$$

For F^2 is invariant under the action of T_a , we have

$$\prod_{i=1}^M A_{ia}^{E_i} = 1, \quad (36)$$

For infinitesimal symmetric operation $(R_a)^\mu_\alpha = \delta_\alpha^\mu + \theta \phi_a^\mu{}_\alpha$, we have

$$\sum_{j=1}^{p_i} \phi_a^{\alpha_j}{}_{\mu_j} G_{\mu_1\ldots\alpha_j\ldots\mu_{p_i}} = A'_{ia} G_{\mu_1\mu_2\ldots\mu_{p_i}}, \quad (37)$$

where $A'_{ia} = \frac{dA_{ia}}{d\theta}|_{\theta=0}$.

We then can find out all of $G_{\mu_1\ldots\mu_{p_i}}$ which satisfy (37) and construct reasonable invariant metric via (36).

A. The de Sitter group

The deformed group of Poincaré group is de Sitter group. As in [10], its natural representation is

$$p_t = \begin{pmatrix} & & & 1 \\ & & 0 & \\ & 0 & & \\ -\lambda & & & \end{pmatrix}, p_x = \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ \lambda & & & 0 \end{pmatrix}, \quad (38)$$

$$p_y = \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ \lambda & & & 0 \end{pmatrix}, p_z = \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ \lambda & & & 0 \end{pmatrix}.$$

The group elements of single parameter Lie group generated by the corresponding generator are

$$\begin{aligned}
 P_t(\theta) &= \begin{pmatrix} \cos(\theta\sqrt{\lambda}) & & \frac{\sin(\theta\sqrt{\lambda})}{\sqrt{\lambda}} \\ & 1 & \\ & & 1 \\ -\sqrt{\lambda}\sin(\theta\sqrt{\lambda}) & & \cos(\theta\sqrt{\lambda}) \end{pmatrix}, \\
 P_x(\theta) &= \begin{pmatrix} 1 & & \frac{\sinh(\theta\sqrt{\lambda})}{\sqrt{\lambda}} \\ \cosh(\theta\sqrt{\lambda}) & & \\ & 1 & \\ \sqrt{\lambda}\sinh(\theta\sqrt{\lambda}) & & \cosh(\theta\sqrt{\lambda}) \end{pmatrix}, \\
 P_y(\theta) &= \begin{pmatrix} 1 & & \frac{\sinh(\theta\sqrt{\lambda})}{\sqrt{\lambda}} \\ & 1 & \\ \cosh(\theta\sqrt{\lambda}) & & \\ & & 1 \\ \sqrt{\lambda}\sinh(\theta\sqrt{\lambda}) & & \cosh(\theta\sqrt{\lambda}) \end{pmatrix}, \\
 P_z(\theta) &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ \cosh(\theta\sqrt{\lambda}) & & \frac{\sinh(\theta\sqrt{\lambda})}{\sqrt{\lambda}} \\ \sqrt{\lambda}\sinh(\theta\sqrt{\lambda}) & & \cosh(\theta\sqrt{\lambda}) \end{pmatrix}.
 \end{aligned} \tag{39}$$

Note that the representation is inherited from the natural representation of Poincaré group in which the representation space has a natural meaning of spacetime and the matrices have the features that the upper left 4×4 part of represents rotation and boost, the upper right 1×4 part represents translation and the lower 5×1 part should keep to be zero. Matrices in (38) indicate that the de Sitter invariant spacetime must be a curved space and the invariant metric is expected to be coordinate dependent. The direct search by (32) or (37) shows that there are invariant tensor neither of rank one or two nor three or four, i.e. the de Sitter invariant metric can not satisfies (??), which is only satisfied by Minkowski manifold.

B. The *DISIM* group

There are two subclasses for *DISIM*, one denoted by *DISIM* in which the *SIM* part is undeformed, while the other in which the *SIM* part is deformed and can be further specified into two different deformation group *XDISIM1* and *XDISIM2*.

1. The *DISIM* group

The deformed generators in *disim* are r_z and b_z with the natural representation matrices,

$$r_z = \begin{pmatrix} A_1 & & & \\ & A_1 & -1 & \\ & 1 & A_1 & \\ & & & A_1 \\ & & & & 0 \end{pmatrix}, b_z = \begin{pmatrix} A_2 & & 1 \\ & A_2 & \\ & & A_2 \\ 1 & & & A_2 \\ & & & & 0 \end{pmatrix}. \tag{40}$$

The corresponding one parameter group elements are,

$$\begin{aligned}
 R_z(\theta) &= e^{A_1\theta} \begin{pmatrix} 1 & & & \\ \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & e^{-A_1\theta} \end{pmatrix}, \\
 B_z(\theta) &= e^{A_2\theta} \begin{pmatrix} \cosh\theta & & \sinh\theta \\ & 1 & \\ \sinh\theta & & \cosh\theta \\ & & 1 & \\ & & & e^{-A_2\theta} \end{pmatrix}.
 \end{aligned} \tag{41}$$

There exists neither rank 1 nor rank 2 invariant tensor. However, there are indeed conformal covariant rank 1 and rank 2 tensors, e.g. the rank 1 tensor, known as spurion,

$$N_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{42}$$

which transforms conformally as $R_z(\theta)(N_\mu) = e^{A_1\theta}N_\mu$ under $R_z(\theta)$ and as $B_z(\theta)(N_\mu) = e^{(1+A_2)\theta}N_\mu$ under $B_z(\theta)$.

The conformally covariant rank 2 tensor under the action of *DISIM* is the Minkowski metric tensor,

$$G_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{43}$$

which transforms as $R_z(\theta)(G_{\mu\nu}) = e^{2A_1\theta}G_{\mu\nu}$ under $R_z(\theta)$ and as $B_z(\theta)(G_{\mu\nu}) = e^{2A_2\theta}G_{\mu\nu}$ under $B_z(\theta)$.

The rank 3 conformally covariant tensor has the form,

$$\begin{aligned}
 F_{t\mu\nu} &= \begin{pmatrix} 3 & & 1 \\ & -1 & \\ & & -1 \\ 1 & & & -1 \end{pmatrix}, F_{x\mu\nu} = \begin{pmatrix} & -1 & \\ -1 & & -1 \\ & & \\ & & -1 \end{pmatrix}, \\
 F_{y\mu\nu} &= \begin{pmatrix} & -1 & \\ -1 & & \\ & & -1 \\ -1 & & & -1 \end{pmatrix}, F_{z\mu\nu} = \begin{pmatrix} 1 & & -1 \\ & -1 & \\ & & -1 \\ -1 & & & -3 \end{pmatrix},
 \end{aligned} \tag{44}$$

which has the conformal factor $e^{3A_1\theta}$ under $R_z(\theta)$ and $e^{(1+3A_2)\theta}$ under $B_z(\theta)$. Actually it is not an independent tensor and can be written as $F_{\sigma\mu\nu} = N_{(\sigma} G_{\mu\nu)}$.

The invariant metric is therefore of the form,

$$F^2 = (N_\mu y^\mu)^A (G_{\mu\nu} y^\mu y^\nu)^B. \quad (45)$$

That the metric function F^2 is a degree 2 homogenous function of y_μ and the invariance under the action of *DISIM*, esp. under $R_z(\theta)$ and under $B_z(\theta)$, gives the constrain condition,

$$\begin{cases} A + 2B = 2 \\ AA_1\theta + 2BA_1\theta = 0 \\ A(1 + A_2)\theta + 2BA_2\theta = 0 \end{cases}, \quad (46)$$

where the first one comes from F^2 as a degree 2 homogenous function, the second one from invariance under $R_z(\theta)$ and the third from $B_z(\theta)$ respectively. The first and the second constrain gives $A_1 = 0$. It means that there does not exist deformed R_z invariant Minkowski-Finsler type of spacetime. Among *DISIM* groups, only those in which R_z is not deformed and only B_z is deformed, denoted by *DISIMb* have the Minkowski-Finsler type of invariant metric. The constrain (46) then becomes,

$$\begin{cases} A + 2B = 2 \\ A(1 + A_2)\theta + 2BA_2\theta = 0 \end{cases}, \quad (47)$$

which has the solution

$$\begin{cases} A = -2A_2 \\ B = 1 + A_2 \end{cases} \quad (48)$$

The *DISIMb* invariant metric function is

$$F^2 = (N_\mu y^\mu)^{-2A_2} (G_{\mu\nu} y^\mu y^\nu)^{1+A_2}, \quad (49)$$

where A_2 is a free parameter which parametrizes the *DISIMb* group.

2. *XDISIM1* and *XDISIM2* groups

We are going to find the invariant metric for *XDISIM1* and *XDISIM2* groups.

The deformed generators in *XDISIM1* are,

$$\begin{aligned} r_z &= \begin{pmatrix} A_2 & & & \\ & A_2 & -1 & \\ & 1 & A_2 & \\ & & & A_2 \\ & & & & 0 \end{pmatrix}, \\ b_z &= \begin{pmatrix} A_3 - A_1 & & & & 1 + A_1 \\ & A_3 - A_1 & & & \\ & & A_3 - A_1 & & \\ & & & A_3 - A_1 & \\ 1 + A_1 & & & & 0 \end{pmatrix}, \\ p_t &= \begin{pmatrix} 0 & & 1 + \frac{A_1}{2} & & \\ & 0 & & & \\ & & 0 & & \\ & & & \frac{A_1}{2} & \\ & & & & 0 \end{pmatrix}, p_z = \begin{pmatrix} 0 & & & & -\frac{A_1}{2} \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 1 + \frac{3A_1}{2} \\ & & & & 0 \end{pmatrix}, \\ p_x &= \begin{pmatrix} 0 & & & & \\ & 0 & & 1 + A_1 & \\ & & & & \\ & & 0 & & \\ & & & 0 & \end{pmatrix}, p_y = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & 1 + A_1 \\ & & & 0 & \\ & & & & 0 \end{pmatrix}. \end{aligned} \quad (50)$$

The conformally covariant rank 1 tensor under the action of *XDISIM1* is still N_μ , which has conformal factor $e^{A_2\theta}$ and $e^{(1+A_3)\theta}$ under $R_z(\theta)$ and $B_z(\theta)$ respectively. The conformally covariant rank 2 tensor is still Minkowski metric tensor $G_{\mu\nu}$, which has conformal factor $e^{2A_2\theta}$ and $e^{2(1+A_3)\theta}$ respectively. So is the case for rank 3 tensor. The metric function is therefore

$$F^2 = (N_\mu y^\mu)^{2\frac{A_1-A_3}{1+A_1}} (G_{\mu\nu} y^\mu y^\nu)^{\frac{1+A_3}{1+A_1}}, \quad (51)$$

which is apparently regresses to the form of *DISIM* when $A_1 = 0$.

The deformed generators for *XDISIM2* are,

$$\begin{aligned} r_z &= \begin{pmatrix} A_2 & & & \\ & A_2 & -1 & \\ & 1 & A_2 & \\ & & & A_2 \\ & & & & 0 \end{pmatrix}, \\ b_z &= \begin{pmatrix} 0 & & & & 1 + 2A_1 - A_3 \\ & A_3 - A_1 & & & \\ & & A_3 - A_1 & & \\ 1 + A_3 & & & 2(A_3 - A_1) & \\ & & & & 0 \end{pmatrix}. \end{aligned} \quad (52)$$

The spurion N_μ is the conformal covariant rank 1 tensor under *XDISIM2*, with conformal factors $e^{A_2\theta}$ and $e^{(A_3-A_1)\theta}$ under $R_z(\theta)$ and $B_z(\theta)$ respectively. The rank 2 conformal covariant tensor has a form rather than $G_{\mu\nu}$

but

$$H_{\mu\nu} = \begin{pmatrix} -\frac{1+A_3}{1+A_1} & & & \frac{A_1-A_3}{1+A_1} \\ & 1 & & \\ & & 1 & \\ \frac{A_1-A_3}{1+A_1} & & \frac{1+2A_1-A_3}{1+A_1} & \end{pmatrix}, \quad (53)$$

with conformal factors $e^{2A_2\theta}$ and $e^{2(A_3-A_1)\theta}$ under $R_z(\theta)$ and $B_z(\theta)$ respectively. What is interesting is that $H_{\mu\nu}$ can not return to $G_{\mu\nu}$ when $A_1 = 0$ but $G_{\mu\nu}$ with a coordinate transformation. Note that the spurion N_μ is still light like with $H_{\mu\nu}$ as the metric tensor,

$$(N_\mu, N_\nu) = H^{\mu\nu} N_\mu N_\nu = 0. \quad (54)$$

The coordinate

$$\begin{cases} t' = t \\ x' = x \\ y' = y \\ z' = z - t \frac{A_1-A_3}{1+2A_1-A_3} \end{cases} \quad (55)$$

can diagonalize the $H_{\mu\nu}$ into

$$H'_{\mu\nu} = \begin{pmatrix} -\frac{1+A_1}{1+2A_1-A_3} & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1+2A_1-A_3}{1+A_1} \end{pmatrix} \quad (56)$$

and transform the spurion into

$$N_\mu = \begin{pmatrix} \frac{1+A_1}{1+2A_1-A_3} \\ \\ \\ 1 \end{pmatrix}. \quad (57)$$

Therefore the *XDISIM2* case can return to the case of *DISIM* by a linear transformation in $t-z$ plane.

Meanwhile, the rank 3 conformal tensor is

$$\begin{aligned} \tilde{F}_{t\mu\nu} &= \begin{pmatrix} 3(1+A_3) & & & 1+3A_3-2A_1 \\ & -(1+A_1) & & \\ & & -(1+A_1) & \\ 1+3A_3-2A_1 & & & -1+3A_3-4A_1 \end{pmatrix}, \\ \tilde{F}_{x\mu\nu} &= \begin{pmatrix} & -(1+A_1) & & \\ -(1+A_1) & & & \\ & & -(1+A_1) & \\ & & & -1+3A_3-4A_1 \end{pmatrix}, \quad \tilde{F}_{y\mu\nu} = \begin{pmatrix} & & -(1+A_1) & \\ & -(1+A_1) & & \\ & & & -(1+A_1) \\ & & & & -1+3A_3-4A_1 \end{pmatrix}, \\ \tilde{F}_{z\mu\nu} &= \begin{pmatrix} 1+3A_3-2A_1 & & & \\ & -(1+A_1) & & \\ & & -(1+A_1) & \\ -1+3A_3-4A_1 & & & -3(1-A_3+2A_1) \end{pmatrix}, \end{aligned} \quad (58)$$

with conformal factors $e^{3A_2\theta}$ and $e^{2(1+3A_3-2A_1)\theta}$ under $R_z(\theta)$ and $B_z(\theta)$ respectively. However it is not an independent tensor for $\tilde{F}_{\sigma\mu\nu} = N_{(\sigma} H_{\mu\nu)}$.

The existence of invariant metric demands no deformation of r_z , i.e. $A_2 = 0$. The invariant metric function is

now

$$F^2 = (N_\mu y^\mu)^2 \frac{A_1-A_3}{1+A_1} (H_{\mu\nu} y^\mu y^\nu)^{\frac{1+A_3}{1+A_1}}. \quad (59)$$

The deformed generators of *XDISIM2* can be expressed with a free parameter,

$$\begin{aligned}
r_z &= \begin{pmatrix} A_2 & & & & \\ & A_2 & -1 & & \\ & 1 & A_2 & & \\ & & & A_2 & \\ & & & & 0 \end{pmatrix}, b_z = \begin{pmatrix} 2(\alpha - A_1) & & & & 1 + 2a - A_3 \\ & A_3 - A_1 & & & \\ & & A_3 - A_1 & & \\ 1 + A_3 + 2(A_1 - \alpha) & & & & 2(A_3 - \alpha) \\ & & & & 0 \end{pmatrix}, \\
p_t &= \begin{pmatrix} 0 & & 1 + \alpha & & \\ & 0 & & & \\ & & 0 & & \\ & & & A_1 - \alpha & \\ & & & & 0 \end{pmatrix}, p_z = \begin{pmatrix} 0 & & \alpha - A_1 & & \\ & 0 & & & \\ & & 0 & 1 + 2A_1 - \alpha & \\ & & & & 0 \end{pmatrix} \\
p_x &= \begin{pmatrix} 0 & & 1 + A_1 & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \end{pmatrix}, p_y = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & 1 + A_1 & \\ & & & & 0 \end{pmatrix}.
\end{aligned} \tag{60}$$

Hence, the conformal covariant rank 1 tensor is spurion N_μ with the conformal factor as in the previous representation. The rank 2 conformal covariant tensor is

$$\tilde{H}_{\mu\nu} = \begin{pmatrix} -\frac{1+A_3+2A_1-2\alpha}{1+A_1} & \frac{2\alpha-A_1-A_3}{1+A_1} & & \\ & 1 & & \\ & & 1 & \\ \frac{2\alpha-A_1-A_3}{1+A_1} & \frac{1+2\alpha-A_3}{1+A_1} & & \end{pmatrix} \tag{61}$$

with conformal factor the same as (53).

3. ISIM group

The rank 1 conformal covariant tensor N_μ for undeformed *SIM* group has conformal factor e^θ under the action of B_z , while the rank 2 conformal covariant tensor $G_{\mu\nu}$ is invariant under B_z . The fully symmetric rank 3 conformal covariant tensor is

$$\begin{aligned}
T_{1\mu\nu} &= \begin{pmatrix} 3a & & a \\ & -a & \\ & & -a \\ a & & -a \end{pmatrix}, T_{2\mu\nu} = \begin{pmatrix} & -a & \\ -a & & -a \\ & -a & \end{pmatrix}, \\
T_{4\mu\nu} &= \begin{pmatrix} a & & -a \\ & -a & \\ & & -a \\ -a & & -3a \end{pmatrix}, T_{3\mu\nu} = \begin{pmatrix} & -a & \\ -a & & -a \\ & -a & \end{pmatrix}.
\end{aligned} \tag{62}$$

However the rank 3 tensor is not independent, it can be decomposed into the direct product of rank 1 and rank 2 tensors. The metric function is therefore of the form

$$F^2 = (N_\mu y^\mu)^A (G_{\mu\nu} y^\mu y^\nu)^B. \tag{63}$$

The invariance of F^2 demands $A = 0$. Finally the metric function is

$$F^2 = G_{\mu\nu} y^\mu y^\nu. \tag{64}$$

C. The DIHOM group

There are two subclasses for *DIHOM*, one subclass denoted by *WDISM* which has the same corresponding structure with *DISIM*, while the other denoted by *DIHOM* which is totally different from *DISIM*.

For the case of *XDISIM* which is lack of r_z , the deformed generator is only b_z . The result is the same as *DISIMb*. Note that the rank 2 conformal covariant tensor has an additional form,

$$F_{\mu\nu} = \begin{pmatrix} & a & b \\ -a & & -a \\ -b & & -b \\ & a & b \end{pmatrix}, \tag{65}$$

which is the only difference needed to be noticed. The tensor is antisymmetric so it is not appropriate to construct invariant metric.

The deformed generators of *DIHOM* are

$$\begin{aligned}
t_2 &= \begin{pmatrix} & & 1 \\ & 1 & & 1 \\ & & -1 & \\ -(A_1 + A_2) & & & -(A_1 + A_2) & 0 \end{pmatrix}, \\
p_t &= \begin{pmatrix} & & & 1 \\ 0 & & 0 & \\ -(A_1 + A_2) & 0 & 0 & \\ & & 0 & 0 \end{pmatrix}, \\
p_x &= \begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ A_1 + A_2 & 0 & & \\ & & 0 & 0 \end{pmatrix}, \\
p_y &= \begin{pmatrix} 0 & & & A_1 \\ & 0 & & \\ & A_1 + A_2 & & 1 \\ A_1 & & 0 & -(A_1 + A_2) \end{pmatrix}, \\
p_z &= \begin{pmatrix} 0 & & & \\ & 0 & & \\ & 0 & A_1 + A_2 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}.
\end{aligned} \tag{66}$$

N_μ is still the rank 1 conformal covariant tensor with conformal factor e^θ and $e^{A_1\theta}$ under the action of B_z and p_y . However, except $N_\mu N_\nu$ the rank 2 conformal covariant tensor for *DIHOM* is only

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} & 1 & \\ -1 & & -1 \\ & & 1 \end{pmatrix}, \tag{67}$$

which is antisymmetric and not appropriate to construct invariant metric. Therefore the invariant metric for *DIHOM* does not exist.

For the undeformed *HOM* group, the rank 3 conformal covariant tensor $T_{\sigma\mu\nu}$ under *SIM* is also conformal covariant under *HOM*. So (63) is also the invariant tensor for *IHOM*.

D. DTE group

There are three classes, where the second and third classes can be specified into two subclasses.

1. DTE1 group

The deformed generator relative to the semidirect of $E(2)$ and $T(4)$ is only r_z ,

$$r_z = \begin{pmatrix} A_1 & & & -A_2 \\ & A_1 + A_2 & -1 & \\ & 1 & A_1 + A_2 & \\ A_2 & & & A_1 + 2A_2 \\ & & & & 0 \end{pmatrix}. \tag{68}$$

The corresponding single parameter group element is

$$R_z(\theta) = e^{(A_1 + A_2)\theta} \begin{pmatrix} 1 - A_2\theta & & -A_2\theta \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \\ A_2\theta & & & 1 + A_2\theta \end{pmatrix}. \tag{69}$$

The spurion N_μ is still the rank 1 conformal covariant tensor of *DTE1* with conformal factor $e^{(A_1 + A_2)\theta}$ under the action of R_z . The rank 2 conformal covariant tensor of *DTE1* is only $N_\mu N_\nu$. Hence the invariant metric under the action of *DTE1* does not exist.

However, in the case of $A_2 = 0$, there exists a kind of rank 2 conformal covariant tensor,

$$G_{\mu\nu} = \begin{pmatrix} a & & b \\ & b - a & \\ b & & b - a \\ & & & 2b - a \end{pmatrix}, \tag{70}$$

with the conformal factor $e^{2A_1\theta}$ under the action of R_z . As in (46) the first and second equation will force $A_1 = 0$ and hence there does not exist *DTE1* invariant metric in all the cases. However there can exist the Minkowski-Finsler type of geometry under the action of *IE(2)*.

2. DTE2 group

DTE2 has two subclasses *DTE2a* and *DTE2b*. *DTE2b* does not have the representation inherited from the 5-d representation of Poincaré group. *DTE2a* has two different kind of natural representations.

In the first natural representation of *DTE2a*, the ma-

trices of four deformed translation generators are

$$\begin{aligned}
 p_t &= \begin{pmatrix} 2\beta - \frac{A_2}{2} & & -\frac{A_2}{2} & 1 \\ & \beta & & \\ -\frac{A_2}{2} & & 2\beta - \frac{A_2}{2} & \\ & & & 0 \end{pmatrix}, \\
 p_x &= \begin{pmatrix} & -\beta & & \\ \beta - A_2 & & \beta - A_2 & 1 \\ & \beta & & \\ & & & 0 \end{pmatrix}, \\
 p_z &= \begin{pmatrix} \frac{A_2}{2} & & \frac{A_2}{2} - 2\beta & \\ & \beta & & \\ 2\beta - \frac{3A_2}{2} & & 4\beta - \frac{3A_2}{2} & 1 \\ & & & 0 \end{pmatrix}, \\
 p_y &= \begin{pmatrix} & -\beta & & \\ \beta - A_2 & & \beta - A_2 & 1 \\ & \beta & & \\ & & & 0 \end{pmatrix},
 \end{aligned} \tag{71}$$

where β satisfies $\beta^2 - A_2\beta + A_1 = 0$.

The spurion N_μ is still the rank 1 conformal covariant tensor here with conformal factor $e^{(2\beta - A_2)\theta}$ and $e^{(2\beta - A_2)\theta}$ under the action of P_t and P_z respectively, while the independent rank 2 conformal covariant tensor rather than $N_\mu N_\nu$ only exists in the case $A_2 = 2\beta$ and is the Minkowski metric tensor $G_{\mu\nu}$ which has conformal factor $e^{(4\beta - A_2)\theta}$ and $e^{2\beta\theta}$ under the action of P_t and P_z respectively. In the case $A_2 = 2\beta$, N_μ is invariant while $G_{\mu\nu}$ is scaled by factor $e^{2\beta\theta}$ under the group action.

The invariant metric under $DTE2a$ in the present representation can only be of the form

$$F = N_\mu y^\mu. \tag{72}$$

In the second natural representation of $DTE2a$, also only translation generators deform with the natural rep-

resentation matrices are

$$\begin{aligned}
 p_t &= \begin{pmatrix} \lambda & & \lambda - A_2 & 1 \\ & \lambda & & \\ \lambda - A_2 & & \lambda & \\ & & & 0 \end{pmatrix}, \\
 p_x &= \begin{pmatrix} & \lambda - A_2 & & \\ \lambda - A_2 & & \lambda - A_2 & 1 \\ & A_2 - \lambda & & \\ & & & 0 \end{pmatrix}, \\
 p_z &= \begin{pmatrix} \lambda & & \lambda - A_2 & \\ & \lambda & & \\ \lambda - A_2 & & \lambda & \\ & & & 0 \end{pmatrix}, \\
 p_y &= \begin{pmatrix} & \lambda - A_2 & & \\ \lambda - A_2 & & \lambda - A_2 & 1 \\ & A_2 - \lambda & & \\ & & & 0 \end{pmatrix},
 \end{aligned} \tag{73}$$

where λ satisfies $\lambda^2 - A_2\lambda + A_1 = 0$.

In the present presentation, the spurion N_μ is still the rank 1 conformal covariant tensor with conformal factor $e^{(2\lambda - A_2)\theta}$ under the action of P_t and P_z . The form of rank 2 conformal covariant tensor under the action $DTE2a$ depends on the λ value. When $\lambda = A_2$, the rank 2 tensor is

$$H_{\mu\nu} = \begin{pmatrix} a & & a+b \\ & b & \\ a+b & & a+2b \end{pmatrix}, \tag{74}$$

while it can only be Minkowski metric tensor $G_{\mu\nu}$ when $\lambda \neq A_2$. The conformal factor is $e^{2\lambda\theta}$ in both cases.

Concerning the construction of the $DTE2a$ invariant metric function, in the case $\lambda \neq A_2$, the invariant metric function has the form

$$F^2 = (G_{\mu\nu} y^\mu y^\nu)^a (N_\mu y^\mu)^b, \tag{75}$$

with the constrain equation

$$\begin{cases} 2a + b = 2 \\ 2a\lambda + b(2\lambda - A_2) = 0 \end{cases}, \tag{76}$$

and solution

$$\begin{cases} a = \frac{A_2 - 2\lambda}{A_2 - \lambda} \\ b = \frac{2\lambda}{A_2 - \lambda} \end{cases}. \tag{77}$$

The metric function is therefore of the form finally

$$F^2 = (G_{\mu\nu} y^\mu y^\nu)^{\frac{A_2 - 2\lambda}{A_2 - \lambda}} (N_\mu y^\mu)^{\frac{2\lambda}{A_2 - \lambda}}. \tag{78}$$

There does not exist $DTE2a$ invariant metric function in the case $\lambda = A_2$.

3. DTE3 group

Like *DTE2*, the *DTE3* can also be specified into two subclasses, *DTE3a* and *DTE3b*. The representation of *DTE3a* is the same as *DTE2a* and hence so is the invariant metric function.

The natural representation of deformed generators in *DTE3b* is

$$\begin{aligned} p_t &= \begin{pmatrix} & -A_1 & A_1 & 1 \\ & & -A_1 & \\ -A_1 & & -2A_1 & \\ & & & 0 \end{pmatrix}, \\ r_z &= \begin{pmatrix} A_2 & & A_2 \\ & -1 & \\ -A_2 & 1 & -A_2 \\ & & & 0 \end{pmatrix}, \\ p_z &= \begin{pmatrix} & -A_1 & A_1 \\ & & -A_1 & \\ -A_1 & & -2A_1 & 1 \\ & & & 0 \end{pmatrix}, \\ p_x &= \begin{pmatrix} & -A_1 & & \\ -A_1 & & -A_1 & 1 \\ & A_1 & & \\ & & & 0 \end{pmatrix}. \end{aligned} \quad (79)$$

The rank 1 conformal covariant tensor is the spurion N_μ with conformal factor $e^{-A_1\theta}$ under the action of P_t and P_z . The rank 2 conformal covariant tensor can only be $N_\mu N_\nu$ in the case of $A_1 \neq 0$. When $A_1 = 0$ it can have different form

$$H_{\mu\nu} = \begin{pmatrix} a & & b \\ & b-a & \\ b & & b-a \\ & & & 2b-a \end{pmatrix}, \quad (80)$$

which is invariant under *DTE3b*.

The construction of the invariant metric function has thus plenty of variety here. The phenomena is similar to *IE(2)* and will be discussed in the invariant metric function of *IE(2)*.

4. IE(2) group

IE(2) group is specific in the aspect that it does not admit only Riemannian structure but also Finslerian structure of spacetime. The rank 2 invariant tensor under

IE(2) is of the form

$$G_{(a,b)\mu\nu} = \begin{pmatrix} a & & a+b \\ & b & \\ a+b & & a+2b \end{pmatrix}, \quad (81)$$

where a and b are free parameters. In case of $b = 0$, it reduces to $N_\mu N_\nu$ while it gives the Minkowski metric tensor in case of $b = -a$. Because of $G_{(a,b)\mu\nu}$ is an invariant rank 2 tensor, the construction of invariant metric function is thus a little bit of arbitrary in the sense that

$$F^2 = \prod_{a,b} (G_{(a,b)\mu\nu} y^\mu y^\nu)^{D_{a,b}}, \quad (82)$$

where it is only need to satisfy the constrain condition

$$\sum_{a,b} D_{a,b} = 1, \quad (83)$$

e.g.

$$F^2 = \frac{(G_{(-1,1)\mu\nu} y^\mu y^\nu)^2}{G_{(1,1)\mu\nu} y^\mu y^\nu} \quad (84)$$

is an admissible form of invariant metric function.

The only *IE(2)* invariant Riemannian metric function is $F^2 = G_{(a,b)\mu\nu} y^\mu y^\nu$ while there are many in Finslerian type.

E. DISO(3) group

There are two classes in classification of deformed group *DISO(3)* of *SO(3)*, *DISO(3)1* which has only one natural representation and *DISO(3)2* which has three inequivalent natural representations.

1. DISO(3)1 group

The matrices of deformed generators of *DISO(3)1* are

$$\begin{aligned} p_t &= \begin{pmatrix} \alpha & & 1 \\ & \alpha & \\ & & \alpha \\ & & & 0 \end{pmatrix}, p_x = \begin{pmatrix} \beta & & 1 \\ \alpha & & 0 \\ & 0 & \\ & & 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} & \beta & \\ 0 & & 1 \\ \alpha & & 0 \\ & 0 & \\ & & 0 \end{pmatrix}, p_z = \begin{pmatrix} & \beta & \\ & 0 & \\ a & & 1 \\ & & & 0 \end{pmatrix}, \end{aligned} \quad (85)$$

where α and β satisfy $\alpha\beta + A_1 = 0$.

There does not exist any rank 1 conformal covariant tensor under the action of $DISO(3)1$. The rank 2 conformal covariant tensor is only Minkowski metric tensor with the conformal factor $e^{2\alpha\theta}$ under the action of P_t . There does not exist $DISO(3)1$ invariant metric function.

2. $DISO(3)2$ group

There are three inequivalent natural representation of $DISO(3)2$ group.

In the first representation, only p_t is deformed

$$p_t = \begin{pmatrix} A_1 & & & 1 \\ & A_1 & & \\ & & A_1 & \\ & & & A_1 \\ & & & & 0 \end{pmatrix}. \quad (86)$$

The rank 1 conformal covariant tensor is not the spurion N_μ anymore but

$$M_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (87)$$

with conformal factor $e^{A_1\theta}$ under the action of P_t . The rank 2 conformal covariant tensor is

$$G_{\mu\nu} = \begin{pmatrix} a & & \\ & b & \\ & & b \\ & & & b \end{pmatrix} \quad (88)$$

with conformal factor $e^{2A_1\theta}$ under the action of P_t . There does not exist invariant metric function under group action.

In the second representation, matrices for deformed generators are

$$\begin{aligned} p_t &= \begin{pmatrix} -A_1 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, p_x = \begin{pmatrix} 0 & & & \\ -A_1 & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 & & & \\ -A_1 & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix}, p_z = \begin{pmatrix} 0 & & & \\ & 0 & & \\ -A_1 & & 0 & 1 \\ & & & 0 \end{pmatrix}. \end{aligned} \quad (89)$$

The rank 1 conformal covariant tensor is M_μ in (87) with conformal factor $e^{-A_1\theta}$ under the action of P_t . There does

not exist any rank 2 conformal covariant tensor. Neither does there the $DISO(3)2$ group invariant metric function.

The same happens to the third representation of $DISO(3)2$ group.

3. $ISO(3)$ group

The $ISO(3)$ invariant rank 1 tensor is M_μ and rank 2 tensor is

$$G_{(a,b)\mu\nu} = \begin{pmatrix} a & & \\ & b & \\ & & b \\ & & & b \end{pmatrix}. \quad (90)$$

The invariant metric function is hence

$$F^2 = (M_\mu y^\mu)^A \prod_{a,b} (G_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}}, \quad (91)$$

with the corresponding constrain condition

$$A + 2 \sum_{a,b} B_{a,b} = 2. \quad (92)$$

F. $DISO(2,1)$ group

There are two classes in the classification of $DISO(2,1)$, $DISO(2,1)1$ which has only one natural representation and $DISO(2,1)2$ which has two inequivalent natural representations.

1. $DISO(2,1)1$ group

The matrices for deformed generators of $DISO(2,1)1$ are

$$\begin{aligned} p_x &= \begin{pmatrix} \alpha & & & \\ & \alpha & & 1 \\ & & \alpha & \\ & & & 0 \end{pmatrix}, p_t = \begin{pmatrix} \alpha & & & 1 \\ \beta & & & \\ & 0 & & \\ & & 0 & \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 & & & \\ & -\beta & & \\ \alpha & & 1 & \\ & & 0 & 0 \end{pmatrix}, p_z = \begin{pmatrix} 0 & & & \\ & -\beta & & \\ & 0 & & 1 \\ \alpha & & & 0 \end{pmatrix}, \end{aligned} \quad (93)$$

where α and β satisfy $\alpha\beta + A_1 = 0$. Like in $DISO(3)1$, we can specify two cases to discuss.

When $A_1 > 0$, we can take $\beta = -\alpha$ and get

$$\begin{aligned} p_x &= \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & 1 \end{pmatrix}, p_t = \begin{pmatrix} & \alpha & & \\ -\alpha & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 & & & \\ & \alpha & & \\ \alpha & & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}, p_z = \begin{pmatrix} 0 & & & \\ & & \alpha & \\ & 0 & & \\ \alpha & & 1 & \\ & & & 0 \end{pmatrix}, \end{aligned} \quad (94)$$

where $\alpha = \pm\sqrt{A_1}$.

There does not exist rank 1 conformal covariant tensor in this case. The conformal covariant rank 2 tensor is just the Minkowski metric tensor.

When $A_1 < 0$, we can take $\beta = \alpha$ and get

$$\begin{aligned} p_x &= \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & 1 \end{pmatrix}, p_t = \begin{pmatrix} & \alpha & & \\ \alpha & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \\ p_y &= \begin{pmatrix} 0 & & & \\ & -\alpha & & \\ \alpha & & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}, p_z = \begin{pmatrix} 0 & & & \\ & & -\alpha & \\ & 0 & & \\ \alpha & & 1 & \\ & & & 0 \end{pmatrix}, \end{aligned} \quad (95)$$

where $\alpha = \pm\sqrt{-A_1}$. There does not exist rank 1 conformal covariant tensor. The conformal covariant rank 2 tensor is

$$H_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \quad (96)$$

In both of two cases, the rank 2 tensor have the conformal factor $e^{2\alpha\theta}$ under P_x and therefore are not appropriate to construct the invariant metric function.

2. $DISO(2,1)2$ group

$DISO(2,1)2$ group has two inequivalent natural representations.

In the first representation, the rank 1 conformal covariant tensor is

$$M_\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (97)$$

with the conformal factor $e^{-A_1\theta}$ under P_x while the rank 2 conformal covariant tensor is

$$H_{\mu\nu} = \begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -a \end{pmatrix}, \quad (98)$$

with the conformal factor $e^{-2A_1\theta}$ under P_x . So this representation does not give invariant metric function.

In the second representation, the rank 1 conformal covariant tensor is still M_μ of (97) with the conformal factor $e^{A_1\theta}$ while there does not exist the rank 2 conformal covariant tensor. So there still does not exist invariant metric function in this representation.

3. $ISO(2,1)$ group

The rank 1 conformal invariant tensor under $ISO(2,1)$ group action is M_μ of (97) while the rank 2 invariant tensor is

$$H_{(a,b)\mu\nu} = \begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -a \end{pmatrix}. \quad (99)$$

The group action invariant metric function is therefore

$$F^2 = (M_\mu y^\mu)^A \prod_{a,b} (H_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}}, \quad (100)$$

with the constrain condition

$$A + 2 \sum_{a,b} B_{a,b} = 2. \quad (101)$$

G. Summary

We summarize the metric functions with respect to various of Poincaré subgroups and the deformed Poincaré subgroups in Table I.

Note that the metric function which is invariant under the deformed Poincaré subgroup is always in the form

$$F^2 = (A_\mu y^\mu)^{2-2 \sum_{a,b} D_{a,b}} \prod_{a,b} (B_{(a,b)\mu\nu} y^\mu y^\nu)^{D_{a,b}}, \quad (102)$$

where A_μ can be one of N_μ , T_μ or X_μ while $B_{(a,b)\mu\nu}$ can be one of $G_{(a,b)\mu\nu}$, $\tilde{G}_{(a,b)\mu\nu}$ or $H_{(a,b)\mu\nu}$ and the combination of A_μ and $B_{(a,b)\mu\nu}$ is different for different group.

TABLE I. The Finsler spacetime metric functions with symmetry group of semi-product Poincaré subgroups and their deformed partners

symmetry	conformal covariant tensor	conformal factor
group	the invariant metric and additional remark	
de Sitter	no conformal covariant tensor and conformal factor	
Poincaré	$G_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	invariant
	$F^2 = G_{\mu\nu} y^\mu y^\nu$	
<i>DISIM</i>	$N_\mu = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}^T$	$B_z(\theta) :$ $e^{(1+A_2)\theta}$
	$G_{\mu\nu}$	$B_z(\theta) :$ $e^{2A_1\theta}$
	$F^2 = (G_{\mu\nu} y^\mu y^\nu)^{1+A_2} (N_\mu y^\mu)^{-2A_2}$	
<i>XDISIM1</i>	N_μ	$B_z(\theta) :$ $e^{(1+A_3)\theta}$
	$G_{\mu\nu}$	$B_z(\theta) :$ $e^{2(A_3-A_1)\theta}$
	$F^2 = (G_{\mu\nu} y^\mu y^\nu)^{\frac{1+A_3}{1+A_1}} (N_\mu y^\mu)^{-2\frac{A_3+A_1}{1+A_1}}$ no invariant metric in case of $A_1 = -1$	
<i>XDISIM2</i>	N_μ	$B_z(\theta) :$ $e^{(1+A_3)\theta}$
	$H_{(M,N)\mu\nu}$	$B_z(\theta) :$
	$M = -\frac{1+A_3}{1+A_1}, N = \frac{A_1-A_3}{1+A_1}$	$e^{2(A_3-A_1)\theta}$
	$F^2 = (H_{(M,N)\mu\nu} y^\mu y^\nu)^{\frac{1+A_3}{1+A_1}} (N_\mu y^\mu)^{-2\frac{A_3-A_1}{1+A_1}}$ a $t-z$ plane non-orthogonal linear transformation is made relative to <i>DISIM</i>	

<i>ISIM</i>	N_μ	$B_z(\theta) : e^\theta$
	$G_{\mu\nu}$	invariant
	$F^2 = G_{\mu\nu} y^\mu y^\nu$	
<i>DIHOM</i>	no invariant metric function	
<i>WDIHOM</i>	the same as <i>DISIM</i>	
<i>IHOM</i>	the same as <i>ISIM</i>	
<i>DTE1</i>	no invariant metric function	
<i>DTE2a1</i>	N_μ	invariant
	$G_{\mu\nu}$	$P_t(\theta), P_z(\theta) :$ $e^{A_2\theta}$
	$F = N_\mu y^\mu$ the relation between A_1 and A_2 : $A_1 = A_2^2/4$	
<i>DTE2a2</i>	N_μ	$P_t(\theta), P_z(\theta) :$ $e^{(2\lambda-A_2)\theta}$
	$G_{\mu\nu}$	$P_t(\theta), P_z(\theta) :$ $e^{2\lambda\theta}$
	$F^2 = (G_{\mu\nu} y^\mu y^\nu)^{\frac{A_2-2\lambda}{A_2-\lambda}} (N_\mu y^\mu)^{\frac{2\lambda}{A_2-\lambda}}$ the deform parameters satisfy: $\lambda^2 - A_2\lambda + A_1 = 0$ and $\lambda \neq A_2$	
<i>DTE2b</i>	no invariant metric function	
<i>DTE3a</i>	the same as <i>DTE2a</i>	
<i>DTE3b</i>	N_μ	invariant
	$H_{(a,b)\mu\nu} = \begin{pmatrix} a & & a+b \\ & b & \\ a+b & & a+2b \end{pmatrix}$	invariant
	$F^2 = \prod_{a,b} (H_{(a,b)\mu\nu} y^\mu y^\nu)^{D_{a,b}}$ the constrain condition: $\sum_{a,b} D_{a,b} = 1$	
<i>TE(2)</i>	the same as <i>DTE3b</i> and hence <i>DTE3b</i> denoted by <i>TE(2)</i>	
<i>DISO(3)1</i>	no invariant metric	
<i>DISO(3)2</i>	no invariant metric	

$ISO(3)$	$T_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$	invariant
	$G_{(a,b)\mu\nu} = \begin{pmatrix} a & & & \\ & b & & \\ & & b & \\ & & & b \end{pmatrix}$	invariant
	$F^2 = (T_\mu y^\mu)^A \prod_{a,b} (G_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}}$ the constrain condition: $A + 2 \sum_{a,b} B_{a,b} = 2$	
$DISO(2,1)1$	no invariant metric	
$DISO(2,1)2$	no invariant metric	
$ISO(2,1)$	$X_\mu = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T$	invariant
	$\tilde{G}_{(a,b)\mu\nu} = \begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -a \end{pmatrix}$	invariant
	$F^2 = (X_\mu y^\mu)^A \prod_{a,b} (\tilde{G}_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}}$ the constrain condition: $A + 2 \sum_{a,b} B_{a,b} = 2$	

The forms frequently appear are

$$F^2 = G_{\mu\nu} y^\mu y^\nu \frac{(N_\mu y^\mu)^2}{(G_{\mu\nu} y^\mu y^\nu)^{1-A} (N_\mu y^\mu)^{2A}}. \quad (103)$$

Among the undeformed groups, only the $ISIM$ group invariant metric function is the Minkowski one while the $TE(2)$, $ISO(3)$ and $ISO(2)$ invariant metric functions are all form of (102), where A_μ is N_μ , T_μ and X_μ and $B_{(a,b)\mu\nu}$ is $H_{(a,b)\mu\nu}$, $G_{(a,b)\mu\nu}$ and $\tilde{G}_{(a,b)\mu\nu}$ respectively.

H. More forms of the metric functions

Invariant metric function form like (102) is representative. However, the invariant metric function can have more plenty of forms available, e.g. the invariant metric function of $DISIM$ is

$$F^2 = (G_{\mu\nu} y^\mu y^\nu)^{1+A_2} (N_\mu y^\mu)^{-2A_2} \quad (104)$$

in general, it is also allowable to have the metric function of the form

$$F^2 = |G_{\mu\nu} y^\mu y^\nu|^{1+A_2} |N_\mu y^\mu|^{-2A_2} \quad (105)$$

TABLE II. invariant zero degree functions of Poincaré subgroup and the deformed partner

symmetric group	invariant zero degree homogenous function ϕ
DISIM	$\phi=1$
XDISIM1	$\phi=1$
XDISIM2	$\phi=1$
ISIM	$\phi=1$
DTE2a1	$\phi=1$
DTE2a2	$\phi=1$
DTE3b	$\phi = \frac{(N_\mu y^\mu)^2}{G_{\mu\nu} y^\mu y^\nu}$
TE(2)	$\phi_{a,b;c,d} = \frac{H_{(a,b)\mu\nu} y^\mu y^\nu}{H_{(c,d)\mu\nu} y^\mu y^\nu}$
ISO(3)	$\phi_{a,b} = \frac{(T_\mu y^\mu)^2}{G_{(a,b)\mu\nu} y^\mu y^\nu}$ and $\phi_{a,b;c,d} = \frac{G_{(a,b)\mu\nu} y^\mu y^\nu}{G_{(c,d)\mu\nu} y^\mu y^\nu}$
ISO(2,1)	$\phi_{a,b} = \frac{(X_\mu y^\mu)^2}{\tilde{G}_{(a,b)\mu\nu} y^\mu y^\nu}$ and $\phi_{a,b;c,d} = \frac{\tilde{G}_{(a,b)\mu\nu} y^\mu y^\nu}{\tilde{G}_{(c,d)\mu\nu} y^\mu y^\nu}$

even that of the form

$$F^2 = \text{sgn}(G_{\mu\nu} y^\mu y^\nu) |G_{\mu\nu} y^\mu y^\nu|^{1+A_2} |N_\mu y^\mu|^{-2A_2}. \quad (106)$$

However, compare to (104) and (105), (106) is better for there may be some region where $G_{\mu\nu} y^\mu y^\nu < 0$ or $N_\mu y^\mu < 0$, (104) is not well defined when the deformation parameter A_2 is not an integer while (105) distinguishes light-like, time-like and space-like totally. (106) is therefor well defined in pseudo-Finsler spacetime.

Moreover, metric function may have plenty of additional structure. If there exist some scalar function $\phi(y^\mu)$, which is invariant under the group action and is the zero degree homogenous function of y^μ , then the product of ϕ and the metric function is still an invariant metric function, e.g.

$$F^2 = G_{\mu\nu} y^\mu y^\nu \cos \left(\omega \frac{T_\mu y^\mu}{X_\mu y^\mu} + \theta \right), \quad (107)$$

where ω and θ are arbitrary parameters. This kind of metric function is allowable in spacetime only possessing rotational symmetry in $y-z$ plane.

To find out the most general form of invariant metric function, one need to determine all such kind of invariant zero degree homogenous function ϕ of y^μ . Table II lists the invariant zero degree homogenous function ϕ of y^μ for various of Poincaré subgroups and their deformed partners.

For DTE3b, TE(2), ISO(3), ISO(2,1), the invariant

metric function can take the form of

$$\left\{ \begin{array}{ll} \text{DTE3b :} & F^2 = (G_{\mu\nu} y^\mu y^\nu)^{1-A} (N_\mu y^\mu)^{2A} \\ & \cdot S(\phi_{\text{DTE3b}}) \\ \text{TE(2) :} & F^2 = \left[\prod_{a,b} (H_{(a,b)\mu\nu} y^\mu y^\nu)^{D_{a,b}} \right] \\ & \cdot S(\phi_{\text{TE(2)}c,d;e,f}) \\ \text{ISO(3) :} & F^2 = \left[\prod_{a,b} (G_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}} \right] \\ & \cdot (T_\mu y^\mu)^A S(\phi_{\text{ISO(3)}c,d}, \phi_{\text{ISO(3)}c,d;e,f}) \\ \text{ISO(2,1) :} & F^2 = \left[\prod_{a,b} (\tilde{G}_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}} \right] \\ & \cdot (X_\mu y^\mu)^A S(\phi_{\text{ISO(2,1)}c,d}, \phi_{\text{ISO(2,1)}c,d;e,f}), \end{array} \right. \quad (108)$$

where S is an arbitrary function.

We thus give all of the Finsler-Minkowski type metric functions which corresponding the 3 generators and four generators Lorentz subgroups and their deformed partner.

Note that the structure of given metric functions is the product of several parts as the ansatz of (28) to enable the group action invariance and the degree 2 homogeneity. Actually, if there are several such groups of components, we can construct metric function by adding different parts constructing from different groups of components together,

$$F = \sum_n F_{(n)}, \quad (109)$$

where $F_{(n)}^n = \sum_i F_i^n$. The Poincaré subgroups in (108) are just those which have more than one group of components to construct the invariant metric function. Therefore they can possess the metric function of the form in (109). For example, $TE(2)$ can have such kind of metric function as

$$F = A \sqrt{G_{\mu\nu} y^\mu y^\nu + (N_\mu y^\mu)^2} + B \sqrt{G_{\mu\nu} y^\mu y^\nu} + C N_\mu y^\mu. \quad (110)$$

I. Conclusion and Outlook

We have obtained all possible group action invariant Finsler-Minkowski metric functions for the semi-product group of three generators and four generators Lorentz subgroups with translation group $T(4)$ and their deformed partner. We find that the group action invariance has strong restrictions on the possible metric functions such that the invariant metric functions for different groups may have the same form. Finally there are only

TABLE III. Various possible symmetric groups and their corresponding invariant metric functions

symmetry group	metric function
DISIM	$F^2 = \text{sgn}(G_{\mu\nu} y^\mu y^\nu) G_{\mu\nu} y^\mu y^\nu ^{1-A} N_\mu y^\mu ^{2A}$ A is given by individual deformation
Poincaré (special case of DISIM)	$F^2 = G_{\mu\nu} y^\mu y^\nu$ the case of $A = 0$
DTE2a1 (special case of DISIM)	$F = N_\mu y^\mu $ the case of $A = 1$
TE(2) (DTE3b)	$F = \sum_N D_N F_{(N)}$ where $F_{(N)}^N = \sum_i C_i F_i^N$ $F_i \in \left\{ f \left f = \sqrt{\prod_{a,b} (H_{(a,b)\mu\nu} y^\mu y^\nu)^{D_{a,b}}} \cdot S(\phi_{\text{TE(2)}c,d;e,f}) \right. \right\}$ $\sum D_{a,b} = 1$
ISO(3)	$F = \sum_N D_N F_{(N)}$ where $F_{(N)}^N = \sum_i C_i F_i^N$ $F_i \in \left\{ f \left f = (T_\mu y^\mu)^A \left[\prod_{a,b} (G_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}} \right] \cdot S(\phi_{\text{ISO(3)}c,d}, \phi_{\text{ISO(3)}c,d;e,f}) \right. \right\}$ $A + 2 \sum B_{a,b} = 2$
ISO(2,1)	$F = \sum_N D_N F_{(N)}$ where $F_{(N)}^N = \sum_i C_i F_i^N$ $F_i \in \left\{ f \left f = (X_\mu y^\mu)^A \left[\prod_{a,b} (\tilde{G}_{(a,b)\mu\nu} y^\mu y^\nu)^{B_{a,b}} \right] \cdot S(\phi_{\text{ISO(2,1)}c,d}, \phi_{\text{ISO(2,1)}c,d;e,f}) \right. \right\}$ $A + 2 \sum B_{a,b} = 2$

several kinds of metric functions corresponding to a few of symmetry groups which is listed in the Table III, where we only list the maximal symmetric group if there are several groups that correspond to the same metric functions.

It can be observed that the undeformed semi-product group usually has richer Finsler structures than the deformed one. The invariant metric corresponding to deformed group usually has larger symmetry. The Finsler metric which corresponds to the largest Poincaré group is unique while the Finsler structure corresponding to

the next to largest symmetry group DISIM is determined uniquely by the deformation parameter of DISIM itself. The semi-product group of three generators Lorentz subgroups have much richer Finsler structure, which can not be determined uniquely even constrained in Riemann geometry. We argue that a reasonable rotation operation should not have the additional accompanied scale transformation, i.e. the Lorentz violation should not happen in the rotation sector but in the boost sector, in our previous paper [10]. The investigation on the invariant metric function indicates that the existence of invariant metric function automatically excludes the additional accompanied scale transformation for rotation operation, i.e. it is a requirement of geometry that the rotation operation is kept even in a Lorentz violation theory.

In our next subsequent paper, we investigate the single particle dynamics and the field theories in the ob-

tained various kind of Finsler-Minkowski spacetime. It reveals that there is the fractional(even irrational) power of derivatives problem both in the single particle dynamics and in the dynamics of field theories. In field theory, we expand the theory according to the power in deformation parameters and result in the lagrangian an non-local log term.

Though the geometries we obtain in this paper have a large freedom, the dynamics in the corresponding spacetime seems supplying constrains on the possible form of geometries. We leave the problem in our next paper.

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